TWISTED K-THEORY AND OBSTRUCTIONS AGAINST POSITIVE SCALAR CURVATURE METRICS

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ABSTRACT. We decompose $\theta(M)$, the twisted index obstruction to a positive scalar curvature metric for closed oriented manifolds with spin universal cover, into a pairing of a twisted K-homology with a twisted K-theory class and prove that $\theta(M)$ does not vanish if M is a closed orientable enlargeable manifold with spin universal cover.

Key Words: enlargeable manifold, index theory, Dirac operator, positive scalar curvature, twisted index obstruction, twisted K-theory

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1. INTRODUCTION

In [22] Rosenberg constructed an index obstruction $\alpha(M) \in KO_n(C^*_{\mathbb{R}}(\pi_1(M)))$ for closed spin manifolds M of dimension n, which vanishes if M admits a metric of positive scalar curvature. It takes values in the K-theory of the (maximal or reduced) real group C^*-algebra associated to the fundamental group and relies on the existence of a spin structure. Gromov, Lawson and Rosenberg conjectured that $\alpha(M)$ is the only obstruction to the existence of a psc metric if $\dim(M) \geq 5$. This was proven to be true in the simply-connected case by Stolz [27], but is false in general as was shown by Schick [24].

Stolz generalized $\alpha(M)$ to the case where M itself may not be spin, but its universal cover $\widetilde{M}$ still is [26], [23, Theorem 1.7]. The new invariant $\theta(M) \in KO(C^*\gamma)$ takes values in the K-theory of a real C^*-algebra associated to a *twisted* version of the fundamental group accounting for the missing spin structure on M. The latter is a $\mathbb{Z}/2\mathbb{Z}$-extension $\widehat{\pi}$ of $\pi = \pi_1(M)$, therefore we will use the notation $C^*(\widehat{\pi} \to \pi)$ instead of $C^*\gamma$.

The element $\alpha(M)$ can be expressed as the pairing of the Dirac class $[D] \in KO_n(M)$ with the KO-theory class $[\mathcal{V}] \in KO_0(C(M, C^*\pi))$ of the Mishchenko-Fomenko bundle. We show that the same is true for $\theta(M)$ if one switches to the *twisted* versions of K-homology and K-theory [7, 2, 14]. For simplicity we treat the complex analogues of the real invariants and refer to [11, Section 1.4] for a nice exposition on how the different indices are related. We identify $\theta(M)$ as the pairing of the twisted fundamental K-homology class $[D^S] \in KK(C(M, \mathbb{C}\ell(M)), \mathbb{C})$ with a twisted version of the Mishchenko-Fomenko bundle representing an element $[\mathcal{V}^S] \in KK(\mathbb{C}, C(M, \mathbb{C}\ell(M)) \otimes C^*(\widehat{\pi} \to \pi))$:

$$\theta(M) = \mathrm{ind}(D_+^{\mathcal{V}}) = [\mathcal{V}^S] \otimes_{C(M,\mathbb{C}\ell(M))} [D^S] \ .$$

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The Dixmier-Douady class of the twist in this case is the element $W_3(M) := \beta(w_2(M)) \in H^3(M;\mathbb{Z})$, i.e. the Bockstein of the second Stiefel-Whitney class. The superscript S denotes the twisted spinor bundle over M. We will use the language of bundle gerbes developed by Murray [19] and the C^*-algebraic versions of their modules [4, 17] to obtain a geometric description of the twisted K-group with coefficients in a C^*-algebra. Most features of index theory are preserved in the twisted case: Murray and Singer proved the analogue of the Atiyah-Singer index theorem in this context [21] and Carey and Wang proved the Thom isomorphism [6] (see also [7]).

As an application of our techniques, we show that $\theta(M)$ does not vanish for closed orientable enlargeable manifolds (Definition 4.1) with spin universal cover and thereby enhance a result of Hanke and Schick [11, 12]:

Theorem 5.7. *Let M be a closed smooth orientable even-dimensional manifold with* $\dim(M) \geq 3$ *and $\widetilde{M}$ spin that is enlargeable in the sense of Definition 4.1. Then we have $\theta^{\max}(M) \neq 0$.*

Extending the suspension argument from [11] it is easy to drop the assumption about even-dimensionality. Gromov and Lawson showed that closed enlargeable manifolds M do not allow a metric of positive scalar curvature. They worked with finite covers in the definition of enlargeability in [9], but later generalized to infinite ones [10]. We also allow the covers $\bar{M} \to M$ to be *non-compact* as in [12]. In this generality, transfer arguments fail, but twisted Hilbert A-module bundles provide a way to circumvent this. This result is independent of the injectivity of the twisted Baum-Connes map.

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Notation. A compact manifold without boundary will be called closed. Throughout the paper, M will denote a smooth closed orientable manifold and A will be a unital C^*-algebra if not stated otherwise. Moreover, $\underline{F}$ will denote a trivial bundle with fiber F if the base space is clear. Whenever we have a surjective submersion $Y \to M$, the notation $Y^{[n]}$ will denote the n-fold fiber product over M.

2. Twisted Hilbert A-module bundles

In this section we will discuss a way to represent classes in twisted K-theory with coefficients in a C^*-algebra A by twisted Hilbert A-module bundles. These are straightforward generalizations of bundle gerbe modules [4, 17]. For an introduction to Hilbert A-modules we refer the reader to [15]. We start by reviewing the geometric description of 2-cocycles via bundle gerbes [19, 20].

Definition 2.1. Let M be a smooth manifold and let $Y \to M$ be a surjective submersion. A *real* line bundle $L \to Y^{[2]}$ will be called a $\mathbb{Z}/2\mathbb{Z}$*-bundle gerbe* (or simply *bundle gerbe* for short) if there exists a multiplication over $Y^{[3]}$, i.e. an isomorphism of line bundles $\mu\colon \pi_{12}^*L \otimes \pi_{23}^*L \to \pi_{13}^*L$, where $\pi_{ij}\colon Y^{[3]} \to Y^{[2]}$ denotes the canonical projections to the fiber product of the ith and jth factor, and such that over $Y^{[4]}$ the following diagram commutes:

$$\begin{tikzcd}
(\pi_{12}^*L \otimes \pi_{23}^*L) \otimes \pi_{34}^*L \arrow[rr, equal] \arrow[d, "\mu \otimes \mathrm{id}"'] & & \pi_{12}^*L \otimes (\pi_{23}^*L \otimes \pi_{34}^*L) \arrow[d, "\mathrm{id} \otimes \mu"] \\
\pi_{13}^*L \otimes \pi_{34}^*L \arrow[r, "\mu"'] & \pi_{14}^*L & \pi_{12}^*L \otimes \pi_{24}^*L \arrow[l, "\mu"]
\end{tikzcd}$$

The bundle gerbe $\delta Q = \pi_1^*Q \otimes \pi_2^*Q^* \to Y^{[2]}$ associated to a line bundle $Q \to Y$ will be called the *trivial bundle gerbe*. Let $L \to Y^{[2]}$ be a bundle gerbe. A choice of a line bundle $Q \to Y$ together with an isomorphism of bundle gerbes $L \to \delta Q$ will be called a *trivialization* of L.

Let $L_i \to Y_i^{[2]}$ for $i \in \{1,2\}$ be two bundle gerbes and let $\pi_i\colon Y_1^{[2]} \times_M Y_2^{[2]} \to Y_i^{[2]}$ denote the projection. The exterior tensor product $L_1 \boxtimes L_2 = \pi_1^*L_1 \otimes \pi_2^*L_2$ is again a bundle gerbe. Each bundle gerbe L has a class $dd(L) \in H^2(M, \mathbb{Z}/2\mathbb{Z})$ canonically associated to it as explained in [17]. $dd(L)$ is called the *Dixmier-Douady class* of L and we summarize its properties in the following theorem proven in [17]:

Theorem 2.2. *Let $L \to Y^{[2]}$ and $L_i \to Y_i^{[2]}$ for $i \in \{1,2\}$ be bundle gerbes. The Dixmier-Douady class has the following properties:*

a) $dd(L) = 0$ if and only if L is isomorphic over $Y^{[2]}$ to a trivial bundle gerbe.

b) $dd(L_1 \boxtimes L_2) = dd(L_1) + dd(L_2)$. □

Definition 2.3. Let L be a bundle gerbe. A covariant derivative $\nabla^L\colon \Omega^0(L) \to \Omega^1(L)$ on L is called a *bundle gerbe connection* if the multiplication isomorphism $\mu\colon \pi_{12}^*L \otimes \pi_{23}^*L \to \pi_{13}^*L$ pulls it back to the canonical connection on the tensor product, i.e.

$$\mu^*\pi_{13}^*\nabla^L = \pi_{12}^*\nabla^L \otimes 1 + 1 \otimes \pi_{23}^*\nabla^L \; . \tag{1}$$

Remark 2.4. The proof for the existence of such connections given in [19] works with the obvious changes for $\mathbb{Z}/2\mathbb{Z}$-bundle gerbes as well. Since the structure group is discrete in our case, every $\mathbb{Z}/2\mathbb{Z}$-bundle gerbe connection is automatically flat.

The main example of bundle gerbes will arise from the following construction.

Definition 2.5. Let Γ be a Lie group (possibly discrete) and let $q\colon \widehat{\Gamma} \to \Gamma$ be a central $\mathbb{Z}/2\mathbb{Z}$-extension of Γ. Let $P \to M$ be a principal Γ-bundle over a manifold M. The line bundle L_P associated to the principal $\mathbb{Z}/2\mathbb{Z}$-bundle $\widehat{L}_P \to P^{[2]}$ with $\widehat{L}_P = \{(p_1, p_2, \widehat{g}) \in P^{[2]} \times \widehat{\Gamma} \mid p_1\, q(\widehat{g}) = p_2\}$ is a bundle gerbe, which will be called the *lifting bundle gerbe* associated to the extension.

For details about this construction see [20, section 6.1], [17, section 2.2]. Since the principal $\mathbb{Z}/2\mathbb{Z}$-bundle associated to any trivialization Q of L_P is a lift of P to a principal $\widehat{\Gamma}$-bundle, we have

Lemma 2.6 ([19])**.** *The class $dd(L_P)$ represents the obstruction lifting $P \to M$ to a principal $\widehat{\Gamma}$-bundle. The isomorphism classes of trivialization of L_P are in $1:1$-correspondence with the possible lifts of P to a principal $\widehat{\Gamma}$-bundle.*

Definition 2.7. Let M be a smooth manifold, A a C^*-algebra and let $Y \to M$ be a surjective submersion. Let $L \to Y^{[2]}$ be a bundle gerbe. A (right) Hilbert A-module bundle $E \to Y$ together with an A-linear action $\gamma\colon L \otimes \pi_2^* E \to \pi_1^* E$ is called a *twisted Hilbert A-module* bundle for L if the following associativity diagram commutes:

$$\begin{tikzcd}
(\pi_{12}^*L \otimes \pi_{23}^*L) \otimes \pi_3^*E \arrow[rr, equal] \arrow[d, "\mu \otimes \mathrm{id}"'] & & \pi_{12}^*L \otimes (\pi_{23}^*L \otimes \pi_3^*E) \arrow[d, "\mathrm{id} \otimes \gamma"] \\
\pi_{13}^*L \otimes \pi_3^*E \arrow[r, "\gamma"'] & \pi_1^*E & \pi_{12}^*L \otimes \pi_2^*E \arrow[l, "\gamma"]
\end{tikzcd}$$

E will be called *finitely generated* and *projective*, if its fibers are finitely generated and projective as Hilbert A-modules. γ will be called the *twisting* of E.

Let E, E' be two twisted Hilbert A-module bundles for the same bundle gerbe L and denote the twistings by γ and γ'. A right A-linear map $f\colon E \to E'$ will be called a *morphism of twisted Hilbert A-module bundles* or (*twisted morphism* for short) if the following diagram commutes:

$$\begin{tikzcd}
L \otimes \pi_2^*E \arrow[r, "\gamma"] \arrow[d, "\mathrm{id}_L \otimes \pi_2^*f"] & \pi_1^*E \arrow[d, "\pi_1^*f"] \\
L \otimes \pi_2^*E' \arrow[r, "\gamma'"] & \pi_1^*E'
\end{tikzcd}$$

For $A = \mathbb{C}$ a twisted Hilbert A-module bundle is the same as a bundle gerbe module [4]. Let $L_i \to Y_i^{[2]}$ for $i \in \{1,2\}$ be two bundle gerbes with $dd(L_1) = dd(L_2)$. A trivialization of $L_1^* \boxtimes L_2$ should be seen as a generalized morphism between L_1 and L_2 [28]. In particular, it induces a push-forward from twisted bundles for L_1 to twisted bundles for L_2 [17, Proposition 2 and 3]:

Lemma 2.8 ([17])**.** *Let $L_i \to Y_i^{[2]}$ for $i \in \{1,2\}$ be two bundle gerbes with $dd(L_1) = dd(L_2)$. Let $Q \to Y_1 \times_M Y_2$ be a trivialization of $L_1^* \boxtimes L_2 \to (Y_1 \times_M Y_2)^{[2]}$. If $E \to Y_1$ is a twisted bundle for L_1 and $\rho\colon Y_1 \times_M Y_2 \to Y_1$ denotes the canonical projection, then $\rho^*E \otimes Q$ descends to a twisted bundle $Q(E) \to Y_2$ for L_2. In particular if $L_2 = \underline{\mathbb{C}}$ over $M^{[2]} = M$, i.e. Q is a trivialization of L_1^*, we obtain an (untwisted) Hilbert A-module bundle $Q(E) \to M$.*

Definition 2.9. Let $L \to Y^{[2]}$ be a bundle gerbe over a surjective submersion $Y \to M$ and let A be a unital C^*-algebra. Denote by $K^0_{L,A}(M)$ the Grothendieck group of isomorphism classes of finitely generated projective twisted Hilbert A-module bundles for L with respect to $\oplus$.

Given a principal $PO(n)$-bundle $P \to M$ there is a bundle of matrix algebras $\mathcal{K} \to M$ with fiber $M_n(\mathbb{C})$ associated to it via the inclusion $PO(n) \to PU(n)$ and the adjoint action of $PU(n)$. Let $L_P \to P^{[2]}$ be the corresponding lifting bundle gerbe for the extension $O(n) \to PO(n)$. We denote a point in the fiber $(L_P)_{(p_1,p_2)}$ by $[\widehat{g},\lambda]$, where $\widehat{g} \in O(n)$ is a lift of the element $g \in PO(n)$ with $p_1\, g = p_2$, $\lambda \in \mathbb{R}$ and $(\widehat{g},-\lambda) \sim (-\widehat{g},\lambda)$ determines $[\widehat{g},\lambda]$.

There is a *canonical twisted vector bundle* $S \to P$ for L_P associated to this construction as follows: Let $S = P \times \mathbb{C}^n$ (see Remark 2.17) with the twisting given by

$$\gamma\colon L_P \otimes \pi_2^* S \to \pi_1^* S \quad ; \quad \gamma([\widehat{g},\lambda] \otimes (p_2,v)) = (p_1, \lambda\, \widehat{g}\, v) \tag{2}$$

Given a twisted Hilbert A-module bundle $E \to Y$ for some bundle gerbe L, note that $S^* \boxtimes E := \pi_P^* S^* \otimes \pi_Y^* E$ over $P \times_M Y$ is a twisted Hilbert A-module bundle for $L_P^* \boxtimes L$.

Theorem 2.10. *Let L be a bundle gerbe, let A be a unital C^*-algebra. Let P, L_P and $\mathcal{K}$ be as in the above paragraph. Assume that $dd(L) = dd(L_P)$ and let Q be a trivialization of $L_P^* \boxtimes L$, then*

$$\kappa_Q\colon K^0_{L,A}(M) \to K_0(C(M,\mathcal{K}) \otimes A) \quad ; \quad [E] \mapsto [C(M, Q(S^* \boxtimes E))]$$

is a well-defined isomorphism.

Proof. Let V be the typical fiber of E. It is a finitely generated projective right Hilbert A-module, therefore $(\mathbb{C}^n)^* \otimes V$ is a finitely generated projective right Hilbert $M_n(\mathbb{C}) \otimes A$-module in a canonical way. Thus, $S^* \boxtimes E$ is a bundle of finitely generated projective right Hilbert $M_n(\mathbb{C}) \otimes A$-modules over $P \times_M Y$. When descending this bundle as in [4, Proposition 4.3], we first obtain bundles $(\mathbb{C}^n)^* \otimes E_i \to U_i$ over an open cover $U_i \subset M$. The transitions over double intersections act via the canonical action of $O(n) \subset U(n)$ on the first tensor factor and in an A-linear way on the second tensor factor. Therefore, $C(M, Q(S^* \boxtimes E))$ carries an action of $C(M, \mathcal{K} \otimes A)$ and the fiberwise $M_n(\mathbb{C}) \otimes A$-valued inner product on $S^* \boxtimes E$ turns $C(M, Q(S^* \boxtimes E))$ into a right Hilbert $C(M, \mathcal{K} \otimes A)$-module. Since V was finitely generated and projective, the same holds true for $C(M, Q(S^* \boxtimes E))$.

Let σ be a projection-valued section of the bundle $\mathcal{K} \otimes M_k(A) \to M$. Let $\pi\colon P \to M$ be the bundle projection, then $\sigma \circ \pi$ yields a section of $\pi^*\mathcal{K} \otimes M_k(A)$. This bundle has a canonical trivialization, such that $\sigma \circ \pi$ yields a continuous map $\widehat{\sigma}\colon P \to M_n(\mathbb{C}) \otimes M_k(A)$ with the property $\widehat{\sigma}(pg) = \widehat{g}^{-1}\, \sigma(p)\, \widehat{g}$. Let $E_\sigma = \{(p,v) \in P \times (\mathbb{C}^n \otimes A^k) \mid \widehat{\sigma}(p)\, v = v\}$. This is a finitely generated, projective twisted Hilbert A-module bundle for L_P. The map $[\sigma] \mapsto [E_\sigma]$ is a homomorphism $K_0(C(M,\mathcal{K}) \otimes A) \to K^0_{L_P,A}(M)$. The composition with the isomorphism $K^0_{L_P,A}(M) \to K^0_{L,A}(M)$ induced by Q^* yields the inverse of the above construction. □

Lemma 2.11. *Consider principal $PO(n_i)$-bundles P_i for $i \in \{1,2\}$ and let $\mathcal{K}_i$ be the associated bundles of complex matrix algebras. We have $dd(L_{P_1}) = dd(L_{P_2})$ if and only if $\mathcal{K}_1 \otimes \mathcal{K}_2 \cong \mathrm{End}(\mathcal{V}) \otimes \mathbb{C}$ for a real vector bundle $\mathcal{V} \to M$. The isomorphism classes of trivializations of $L_{P_1}^* \boxtimes L_{P_2}$ are in $1:1$-correspondence with the isomorphism classes of possible $\mathcal{V}$'s.*

Proof. Let $O(n_1) \otimes O(n_2)$ be the quotient of the product $O(n_1) \times O(n_2)$ by the diagonal $Z/2\mathbb{Z}$-action. If $dd(L_{P_1}) = dd(L_{P_2})$, then $P_1 \times_M P_2$ lifts to a principal $O(n_1) \otimes O(n_2)$-bundle P and $\mathcal{V} = P \times_\rho (\mathbb{R}^{n_1} \otimes \mathbb{R}^{n_2})$ satisfies the condition, where ρ is the standard representation. The isomorphism class of $\mathcal{V}$ only depends on the one of P and the choices of possible P's are in $1:1$-correspondence with the trivializations of $L_{P_1}^* \boxtimes L_{P_2}$.

On the other hand, if $\mathcal{V}$ exists, let P be its principal $O(n_1 n_2)$-bundle. $P/(\mathbb{Z}/2\mathbb{Z})$ reduces to the principal $PO(n_1) \times PO(n_2)$-bundle $P_1 \times_M P_2$ and the pullback of $P \to P/(\mathbb{Z}/2\mathbb{Z})$ via $P_1 \times_M P_2 \to P/(\mathbb{Z}/2\mathbb{Z})$ is a lift of $P_1 \times_M P_2$ to a principal $O(n_1) \otimes O(n_2)$-bundle, which shows that $dd(L_{P_1}) = dd(L_{P_2})$. □

Remark 2.12. Note that $C(M, \mathcal{V} \otimes \mathbb{C}) \otimes A$ provides a Morita equivalence bimodule between $C(M, \mathcal{K}_1) \otimes A$ and $C(M, \mathcal{K}_2) \otimes A$ and $\psi_{\mathcal{V}} \colon K_0(C(M, \mathcal{K}_1) \otimes A) \to K_0(C(M, \mathcal{K}_2) \otimes A)$ with $\psi_{\mathcal{V}}([X]) = [X \otimes_{C(M,\mathcal{K}_1)\otimes A} C(M, \mathcal{V} \otimes \mathbb{C}) \otimes A]$ is the corresponding isomorphism on K-theory.

Let L be a bundle gerbe. Let $\mathcal{K}_i$, P_i for $i \in \{1, 2\}$ be as in Lemma 2.11. Suppose that $dd(L) = dd(L_{P_1}) = dd(L_{P_2})$. Trivializations Q_i of $L^* \boxtimes L_{P_i}$ induce a trivialization of $L_{P_1}^* \boxtimes L_{P_2}$, since $Q_1^* \boxtimes Q_2$ over $P_1 \times_M P \times_M P \times_M P_2$ descends to a line bundle Q_{12} over $P_1 \times_M P_2$ in a way compatible with the actions of L_{P_i}. The trivialization Q_{12} corresponds to a real vector bundle $\mathcal{V}_{12}$ as described in Lemma 2.11. Let $\kappa_{Q_i}([E]) = [C(M, Q_i(S_i^* \boxtimes E))]$ be the homomorphism from Theorem 2.10 associated to the data Q_i and the canonical bundle S_i. A bundle gerbe L provides a canonical trivialization of $L^* \boxtimes L$ (in particular, if P is a principal $PO(n)$-bundle, then the principal $PO(n) \times PO(n)$-bundle $P^{[2]}$ has a canonical lift to a $O(n) \otimes O(n)$-bundle). Let $\kappa_{L_{P_i}}$ be the associated isomorphism from Theorem 2.10. Then the the following diagram of isomorphisms commutes:

$$\begin{tikzcd}
 & K_0(C(M,\mathcal{K}_1)\otimes A) \arrow[dd, "\psi_{\mathcal{V}}"] & K^0_{L_{P_1},A}(M) \arrow[l, "\kappa_{L_{P_1}}"'] \arrow[dd, "Q_{12}"] \\
K^0_{L,A}(M) \arrow[ur, "\kappa_{Q_1}"] \arrow[dr, "\kappa_{Q_2}"'] & & \\
 & K_0(C(M,\mathcal{K}_2)\otimes A) & K^0_{L_{P_2},A}(M) \arrow[l, "\kappa_{L_{P_2}}"]
\end{tikzcd}$$

2.1. **The twisted Mishchenko-Fomenko bundle.** The obstruction $\alpha(M)$ is built out of the Dirac operator on M and a certain bundle of Hilbert $C^*(\pi_1(M))$-modules called the Mishchenko-Fomenko bundle. In the twisted case the algebra as well as the bundle itself have to be replaced by twisted versions.

Definition 2.13. Let $1 \to \mathbb{Z}/2\mathbb{Z} \to \widehat{\pi} \to \pi \to 1$ be a central $\mathbb{Z}/2\mathbb{Z}$-extension of a discrete group π. Let $C_r^*(\widehat{\pi})$ be the reduced group C^*-algebra associated to $\widehat{\pi}$, likewise let $C^*_{\max}(\widehat{\pi})$ be its maximal group C^*-algebra. Let $e \in \mathbb{Z}/2\mathbb{Z}$ be the non-trivial element and observe that $q = \frac{1}{2}(1-e) \in C^*_{r/\max}(\widehat{\pi})$ is a central projection. Let $C_r^*(\widehat{\pi} \to \pi) = q\, C_r^*(\widehat{\pi}) = q\, C_r^*(\widehat{\pi})\, q$ and define $C^*_{\max}(\widehat{\pi} \to \pi)$ analogously. It is straightforward to check that these are in fact C^*-algebras, which we will call the *reduced*, respectively *maximal twisted group C^*-algebra* [26, Definition 8.1].

Remark 2.14. An extension in the above definition can be classified by a group 2-cocycle $c_{\widehat{\pi}} \in H^2_{\mathrm{gr}}(\pi, \mathbb{Z}/2\mathbb{Z})$, which yields an alternative description of $C_r^*(\widehat{\pi} \to \pi)$ using completions of the twisted group ring $\mathbb{C}[\pi, c_{\widehat{\pi}}]$ [5, Definition 2.1 for $A = \mathbb{C}$]. The algebra $C^*_{\max}(\widehat{\pi} \to \pi)$ has the universal property that any projective representation $\rho\colon \pi \to U(H)$ for the cocycle $c_{\widehat{\pi}}$ extends to a $*$-homomorphism $C^*_{\max}(\widehat{\pi} \to \pi) \to B(H)$.

Let M be a closed smooth oriented manifold with $\dim(M) \geq 3$, such that its universal cover $\widetilde{M}$ carries a spin structure. Denote by P_{SO} the oriented frame bundle of M and let $\pi = \pi_1(M)$. Let $P_{SO(\widetilde{M})}$ be the frame bundle of the universal cover. Consider the exact sequence

$$\pi_2(\widetilde{M}) \to \pi_1(SO(n)) \to \pi_1(P_{SO(\widetilde{M})}) \to 1 \tag{3}$$

The map $\pi_2(\widetilde{M}) \to \pi_1(SO(n))$ sends $f\colon S^2 \to \widetilde{M}$ to the transition map $\varphi_f\colon S^1 \to SO(n)$ obtained from the pullback $f^*P_{SO(\widetilde{M})}$. If $\widetilde{M}$ carries a spin structure, then φ_f factors through $S^1 \to \mathrm{Spin}(n)$ and is therefore nullhomotopic. Comparing (3) with the exact sequence

$$\pi_2(M) \to \pi_1(SO(n)) \to \pi_1(P_{SO}) \to \pi_1(M) \to 1$$

for the bundle $P_{SO} \to M$, a diagram chase shows that $\pi_2(M) \to \pi_1(SO(n))$ is also trivial in this case and we obtain a central $\mathbb{Z}/2\mathbb{Z}$-extension [26]:

$$1 \to \mathbb{Z}/2\mathbb{Z} \to \pi_1(P_{SO}) \to \pi \to 1\ . \tag{4}$$

Definition 2.15. The lifting bundle gerbe $L_{\widehat{\pi}} \to \widetilde{M}^{[2]}$ associated to the above central extension (4) is called the *Mishchenko-Fomenko bundle gerbe*.

Note that a trivialization of $L_{\widehat{\pi}} \to \widetilde{M}^{[2]}$ *as a line bundle* is given by a split $\pi \to \pi_1(P_{SO}) = \widehat{\pi}$. After applying this trivialization, the bundle gerbe multiplication over points $\tilde{m}_1, \tilde{m}_2, \tilde{m}_3$ with $\tilde{m}_2 = \tilde{m}_1 g_1$, $\tilde{m}_3 = \tilde{m}_2 g_2$ for some $g_1, g_2 \in \pi$ is transformed to $\mu((\tilde{m}_1, \tilde{m}_2, z_1), (\tilde{m}_2, \tilde{m}_3, z_2)) = (\tilde{m}_1, \tilde{m}_3, z_1\, z_2\, c_{\widehat{\pi}}(g_1, g_2))$ for some cocycle $c_{\widehat{\pi}}\colon \pi \times \pi \to \mathbb{Z}/2\mathbb{Z}$ representing the extension in the group $H^2(\pi, \mathbb{Z}/2\mathbb{Z})$ and $z_i \in \mathbb{R}$.

Definition 2.16. Let $\widehat{\pi} = \pi_1(P_{SO})$ and let $1 \to \mathbb{Z}/2\mathbb{Z} \to \widehat{\pi} \to \pi \to 1$ be the extension described above. The canonical twisted Hilbert A-module bundle for $A = C^*_{\max}(\widehat{\pi} \to \pi)$ given by $\mathcal{V}_{\max} = \widetilde{M} \times C^*_{\max}(\widehat{\pi} \to \pi)$ is called the *(maximal) twisted Mishchenko-Fomenko bundle*. Similarly, we define $\mathcal{V}_{\mathrm{red}} = \widetilde{M} \times C^*_r(\widehat{\pi} \to \pi)$. In both cases the twisting is given as in (2).

Remark 2.17. Although $\mathcal{V}_{\mathrm{red}}$ is trivial as a bundle over $\widetilde{M}$, the twisting provides extra structure that replaces the associated bundle construction, which does not make sense in the twisted case. This is analogous to viewing a vector bundle E with fiber V and frame bundle P_{SO} as the equivariant vector bundle $P_{SO} \times V$ over P_{SO} and keeping track of the action.

Let $E \to \widetilde{M}$ be a twisted Hilbert A-module bundle for $L_{\widehat{\pi}}$. Trivializing the bundle gerbe $L_{\widehat{\pi}}$ (and thereby fixing a cocycle $c_{\widehat{\pi}}$) turns the twisting into a collection of bundle maps that are on the fibers given by $\gamma^g\colon E_{\tilde{m}} \to E_{\tilde{m}g^{-1}}$ with the property that $\gamma^g \circ \gamma^h = c_{\widehat{\pi}}(g,h)\gamma^{gh}$, i.e. it behaves like a projective representation.

Now suppose that E is smooth and comes equipped with a twisted connection ∇. Fix $\widetilde{m} \in \widetilde{M}$, let m be its image in M and let $c\colon I \to \widetilde{M}$ be a smooth curve in $\widetilde{M}$ starting at $\widetilde{m}$. Let $\mathcal{P}_c\colon I \times E_{\widetilde{m}} \to E$ be the parallel transport with respect to ∇. Let $\widehat{g} \in \widehat{\pi}$, let $g \in \pi$ be its image under the projection. Observe that $\mathcal{P}_c$ is equivariant in the sense that $\mathcal{P}_{cg^{-1}}(t, \widehat{g} \cdot v) = \widehat{g} \cdot \mathcal{P}_c(t,v)$ for $t \in I$, $v \in E$. Let $\tau\colon S^1 \to M$ be a smooth loop in M at m representing an element $h \in \pi = \pi_1(M)$, let $\bar{\tau}\colon I \to \widetilde{M}$ be the lift of τ to $\widetilde{m}$. Note that the endpoint of $\bar{\tau}$ is $\widetilde{m}h^{-1}$. Define $\mathcal{P}(\widetilde{m}, \tau)(v) := \mathcal{P}_{\bar{\tau}}(1, v)$ and let $\mathrm{hol}(\tau, \widetilde{m})\colon E_{\widetilde{m}} \to E_{\widetilde{m}}$ be given by $\mathrm{hol}(\tau, \widetilde{m}) = \gamma^{h^{-1}} \circ \mathcal{P}(\widetilde{m}, \tau)$. It satisfies

$$\begin{aligned} c_{\widehat{\pi}}(g,h)^{-1}\,\mathrm{hol}(\bar{m}, \sigma * \tau) &= c_{\widehat{\pi}}(g,h)^{-1}\gamma^{(gh)^{-1}}\mathcal{P}(\bar{m}, \sigma * \tau) = \gamma^{h^{-1}} \circ \gamma^{g^{-1}} \circ \mathcal{P}(\bar{m}g^{-1}, \tau) \circ \mathcal{P}(\bar{m}, \sigma) \\ &= \gamma^{h^{-1}} \circ \left(\gamma^{g^{-1}} \circ \mathcal{P}(\bar{m}g^{-1}, \tau) \circ \gamma^{g}\right) \circ \gamma^{g^{-1}} \circ \mathcal{P}(\bar{m}, \sigma) \\ &= \gamma^{h^{-1}} \circ \mathcal{P}(\bar{m}, \tau) \circ \gamma^{g^{-1}} \circ \mathcal{P}(\bar{m}, \sigma) = \mathrm{hol}(\bar{m}, \tau) \circ \mathrm{hol}(\bar{m}, \sigma)\ , \end{aligned}$$

which proves that it is a projective representation of π on $E_{\widetilde{m}}$ with respect to $c_{\widehat{\pi}}$.

Definition 2.18. We will call $\mathrm{hol}(\cdot, \widetilde{m})$ the *projective holonomy of* ∇ *at* $\widetilde{m}$.

3. Index Theory on Twisted Bundles

3.1. **The Chern Character.** Twisted versions of the Chern character have been developed in [4, Section 6], [21, Section 3.2]. Crucial in both cases is the idea that the endomorphism bundle $\mathrm{End}(F) \to Y$ of a twisted vector bundle F for a bundle gerbe L descends to a vector bundle $\mathrm{end}(F)$ over M. Let ∇^F be a twisted connection on F. Since we deal with the case of real bundle gerbes, where curving forms do not matter, the curvature descends to a form $\Omega_F \in \Omega^2(M, \mathrm{end}(F))$.

Definition 3.1. The *twisted Chern character* $\mathrm{ch}(F)$ is the class

$$\mathrm{ch}(F) = \mathrm{tr}\left(\exp\left(\frac{i\Omega_F}{2\pi}\right)\right) \in H^{\mathrm{even}}(M, \mathbb{R})\ . \tag{5}$$

It induces a group homomorphism $\mathrm{ch}\colon K^0_{L,\mathbb{C}}(M) \to H^{\mathrm{even}}(M, \mathbb{R})$. If L_1, L_2 are two bundle gerbes with $dd(L_1) = dd(L_2)$ and Q is a trivialization of $L_1^* \boxtimes L_2$, then

$$\begin{tikzcd} K^0_{L_1,\mathbb{C}}(M) \arrow[dr, "\mathrm{ch}"] \arrow[dd, "Q"'] & \\ & H^{\mathrm{even}}(M, \mathbb{R}) \\ K^0_{L_2,\mathbb{C}}(M) \arrow[ur, "\mathrm{ch}"'] & \end{tikzcd}$$

commutes, since Q is a flat line bundle.

Choose a lifting bundle gerbe L_P with $dd(L) = dd(L_P)$. The Künneth theorem and the canonical isomorphism $K^0_{L_P,\mathbb{C}}(M) \to K^0(C(M, \mathcal{K}))$ allows us to extend the above definition to a $K_0(A) \otimes \mathbb{R}$-valued Chern character via

$$\mathrm{ch}\colon K^0_{L,A}(M) \to K^0_{L_P,\mathbb{C}}(M) \otimes K_0(A) \otimes \mathbb{R} \to H^{\mathrm{even}}(M, K_0(A) \otimes \mathbb{R})\ .$$

The first map is obtained from Theorem 2.10 using a trivialization Q of $L^* \boxtimes L_P$. As we have seen in Remark 2.12, the result of ch does not depend on the choice of P and L_P.

Lemma 3.2. *Let $L_i \to Y_i^{[2]}$ for $i \in \{1,2\}$ be bundle gerbes. The following diagram commutes:*

$$\begin{tikzcd} K^0_{L_1,A}(M) \times K^0_{L_2,\mathbb{C}}(M) \arrow[r, "\boxtimes"] \arrow[d, "\mathrm{ch}\times\mathrm{ch}"] & K^0_{L_1\boxtimes L_2,A}(M) \arrow[d, "\mathrm{ch}"] \\ H^{\mathrm{even}}(M, K_0(A)\otimes\mathbb{R}) \times H^{\mathrm{even}}(M,\mathbb{R}) \arrow[r, "\wedge"] & H^{\mathrm{even}}(M, K_0(A)\otimes\mathbb{R}) \end{tikzcd}$$

Proof. For $A = \mathbb{C}$ the proof is analogous to the untwisted case. For arbitrary A, choose a principal $PO(n)$-bundle P and a trivialization Q of $L_1^* \boxtimes L_P$ as in Theorem 2.10. Note that Q and the canonical trivialization of $L_2^* \boxtimes L_2$ induce a trivialization of $(L_1 \boxtimes L_2)^* \boxtimes (L_P \boxtimes L_2)$. The statement is a consequence of the following commutative diagram

$$\begin{tikzcd} K^0_{L_1,A}(M) \times K^0_{L_2,\mathbb{C}}(M) \arrow[r] \arrow[d] & K^0_{L_P,\mathbb{C}}(M) \otimes K_0(A) \otimes \mathbb{R} \times K^0_{L_2,\mathbb{C}}(M) \arrow[d] \\ K^0_{L_1\boxtimes L_2,A}(M) \arrow[r] & K^0_{L_P\boxtimes L_2,\mathbb{C}}(M) \otimes K_0(A) \otimes \mathbb{R} \end{tikzcd}$$

in which the vertical maps are given by tensor products and the horizontal ones are the decomposition maps from the Künneth theorem. □

3.1.1. *Chern character and traces.* If A comes equipped with a trace τ, which we will assume for the rest of this section, there is a more direct approach to the Chern character, which was explored in the untwisted case in [25, Section 4]. If V is a finitely generated projective Hilbert A-module, then we have $\mathrm{End}(V) = \mathcal{K}(V,V) \cong V \otimes_A \mathcal{K}(V,A)$, where $\mathcal{K}(V,W)$ denotes the compact adjointable A-linear operators. Setting $\tau_V(v \otimes T) = \tau(T(v))$ on elementary tensors, extends the trace to $\mathrm{End}(V)$.

Let L be a bundle gerbe and E be a finitely generated projective twisted Hilbert A-module bundle for L. As was already noted above, $\mathrm{End}(E)$ descends to a bundle of C^*-algebras $\mathrm{end}(E)$ over M. Extending τ to the fibers yields a linear map $\Omega^n(M, \mathrm{end}(E)) \to \Omega^n(M)$ on $\mathrm{end}(E)$-valued forms, which we will again denote by τ.

Definition 3.3. Let M be a closed smooth manifold, L be a bundle gerbe and E be a twisted Hilbert A-module bundle for L equipped with a twisted connection ∇^E that has curvature $\Omega_E \in \Omega^2(M, \mathrm{end}(E))$. As in [25, Lemma 4.2] it follows that

$$\tau\left(\exp\left(\frac{i\Omega_E}{2\pi}\right)\right) \in \Omega^{\mathrm{even}}(M)$$

is a closed form, which is independent of the choice of twisted connection. Its cohomology class $\mathrm{ch}_\tau(E)$ is called the *τ-Chern character* of E.

To explain the connection between ch from Definition 3.1 and ch_τ we need the following concept.

Definition 3.4. Let V be a finitely generated projective right Hilbert A-module. Its *dimension* is defined to be $\dim_\tau(V) = \tau_V(\mathrm{id}_V)$, where τ_V. If $V \cong tA^n$ for some projection $t \in M_n(A) = M_n(\mathbb{C}) \otimes A$, then $\dim_\tau(V) = (\mathrm{tr} \otimes \tau)(t)$. $\dim_\tau$ yields a well-defined map $K_0(A) \to \mathbb{R}$.

Theorem 3.5. *Let L be a bundle gerbe and let E be a twisted Hilbert A-module bundle for L. Then we have*

$$\dim_\tau(\mathrm{ch}(E)) = \mathrm{ch}_\tau(E) \ \in H^{even}(M,\mathbb{R}) \ . \tag{6}$$

Proof. First observe that ch_τ is still multiplicative, in the sense that for a twisted Hilbert A-module bundle E and a twisted vector bundle F we have $\mathrm{ch}_\tau(E \boxtimes F) = \mathrm{ch}_\tau(E) \cup \mathrm{ch}(F)$. In fact, this property rests only on the fact that the canonical tensor product connection $\nabla^{E\boxtimes F}$ has curvature $\Omega_{E\boxtimes F} = \Omega_E \otimes \mathrm{id} + \mathrm{id} \otimes \Omega_F \in \Omega^2(M, \mathrm{end}(E) \otimes \mathrm{end}(F))$ together with the calculation

$$\tau\left(\exp\left(\frac{i\,\Omega_{E\boxtimes F}}{2\pi}\right)\right) = \tau\left(\exp\left(\frac{i\,\Omega_E}{2\pi}\right)\right) \wedge \tau\left(\exp\left(\frac{i\,\Omega_F}{2\pi}\right)\right)$$

which is proven as in the untwisted case.

Choose P and a trivialization Q as in Theorem 2.10 and let F be a twisted vector bundle for L_P. Let V be a finitely generated projective Hilbert A-module. Observe that for the trivial bundle $\underline{V}$ we have $\mathrm{ch}_\tau(\underline{V}) = \dim_\tau(V)$. Thus,

$$\mathrm{ch}_\tau(F \otimes V) = \mathrm{ch}_\tau(F \boxtimes \underline{V}) = \mathrm{ch}(F) \cup \mathrm{ch}_\tau(\underline{V}) = \dim_\tau(V)\ \mathrm{ch}(F)\ .$$

Likewise we have $\mathrm{ch}(F\otimes V) = \mathrm{ch}(F)\otimes[V] \in H^{\mathrm{even}}(M, K_0(A)\otimes\mathbb{R})$. Therefore the following diagram commutes

$$\begin{tikzcd}
K^0_{L_P,\mathbb{C}}(M) \otimes K_0(A) \arrow[r] \arrow[dr, "\mathrm{ch}\otimes\dim_\tau"'] & K_0(C(M,\mathcal{K}\otimes A)) \arrow[r, "\rho_Q^{-1}"] & K^0_{L,A}(M) \arrow[dl, "\mathrm{ch}_\tau"] \\
& H^{\mathrm{even}}(M,\mathbb{R}) &
\end{tikzcd}$$

where the first horizontal map is given by the canonical identification $K^0_{L_P,\mathbb{C}}(M) \cong K_0(C(M,\mathcal{K}))$ and the tensor product of modules. The statement follows if ch_τ vanishes on the second summand in the Künneth decomposition. We can identify $K^1_{L_P,\mathbb{C}}(M)$ with $K^0_{\pi^*L_P,\mathbb{C}}(\mathbb{R}\times M)$. Likewise we can represent classes in $K_1(A)$ as compactly supported virtual Hilbert A-module bundles over $\mathbb{R}$. If $F = (F_+, F_-, \varphi)$ represents an element in $K^0_{\pi^*L_P,\mathbb{C}}(\mathbb{R}\times M)$ and $W = (W_+, W_-, \psi)$ an element in $K_0(C_0(\mathbb{R}, A))$, then the graded tensor product $F \mathbin{\widehat{\boxtimes}} W$ yields an element in $K^0_{\pi^*L_P,A}(\mathbb{R}^2\times M)$ and Bott periodicity maps it to $b([F \mathbin{\widehat{\boxtimes}} W]) \in K^0_{L_P,A}(M)$. It is a consequence of the multiplicativity that $\mathrm{ch}_\tau(X) = \mathrm{ch}_\tau(b(X)) \cup e$, where $e \in H^2_c(\mathbb{R}^2,\mathbb{R}) \cong \mathbb{R}$ denotes a generator and X is a triple representing a class in $K^0_{\pi^* L_P, A}(\mathbb{R}^2 \times M)$. Thus,

$$\mathrm{ch}_\tau(b(F \boxtimes W)) \cup e = \mathrm{ch}_\tau(F \boxtimes W) = \mathrm{ch}(F) \cup \mathrm{ch}_\tau(W) = 0$$

since $\mathrm{ch}_\tau(W) \in H^{\mathrm{even}}_c(\mathbb{R}, \mathbb{R}) = 0$. □

3.2. **Local C^*-algebras and twisted bundles.** A *local C^*-algebra* is a $*$-algebra B equipped with a pre-C^*-norm such that $M_n(B)$ is closed under holomorphic functional calculus for all $n \in \mathbb{N}$. If $\mathcal{B} \to M$ is a locally trivial bundle with fiber B and structure group $\mathrm{Aut}(B)$ equipped with the point norm topology, then the section algebra $C(M, \mathcal{B})$ is again a local C^*-algebra as can be checked directly from the definitions using the naturality of holomorphic functional calculus.

A right B-module is called *finitely generated* and *projective* if it is isomorphic to one of the form $t\,B^n$ for a projection $t \in M_n(B)$. The groups $K_i(B)$, $i \in \{0, 1\}$ are now defined as in [3]. If A is the completion of B, then $K_i(B) \cong K_i(A)$. Twisted B-module bundles are defined just as in 2.7 and yield analogous groups $K^0_{L,B}(M)$ and $K^1_{L,B}(M) := K^0_{\pi_M^* L, B}(\mathbb{R} \times M)$. Given a twisted B-module bundle $\mathcal{E}$, the fiberwise inner product $\mathcal{E} \otimes_B A$ is a twisted Hilbert A-module bundle. Thus we get a homomorphism $K^0_{L,B}(M) \to K^0_{L,A}(M)$

Let S be as in Theorem 2.10. The exterior tensor product $S^* \boxtimes \mathcal{E}$ involves no completion. Therefore the proof of that theorem still works and gives $K^0_{L,B}(M) \cong K_0(C(M, \mathcal{K} \otimes B))$, which also proves $K^0_{L,B}(M) \cong K^0_{L,A}(M)$. The Bott map $b\colon K^0_{L,B}(M) \to K^0_{\pi_M^* L, B}(\mathbb{R}^2 \times M)$ is also still well-defined and a comparison with the corresponding map for its completion shows that b still is an isomorphism. Likewise, there is a Künneth homomorphism

$$K^0_{L_P, \mathbb{C}}(M) \otimes K_0(B) \otimes \mathbb{R} \oplus K^1_{L_P, \mathbb{C}}(M) \otimes K_1(B) \otimes \mathbb{R} \to K_0(C(M, \mathcal{K} \otimes B)) \otimes \mathbb{R}$$

that just involves algebraic tensor products in the fibers. Comparison with the completion reveals this to be an isomorphism as well. This enables us to define $\mathrm{ch}\colon K^0_{L,B}(M) \to H^{\mathrm{even}}(M, K_0(B) \otimes \mathbb{R})$.

Let B be a unital local C^*-algebra carrying a trace τ. If $\mathcal{V}$ is a finitely generated and projective B-module, then $\mathrm{End}(\mathcal{V})$ coincides with the finite rank operators and we have $\mathrm{End}(\mathcal{V}) \cong \mathcal{V} \otimes_B \mathrm{Hom}(\mathcal{V}, B)$, where the tensor product is a quotient of the algebraic tensor product. Repeating the constructions in Section 3.1.1 we can define $\mathrm{ch}_\tau\colon K^0_{L,B}(M) \to H^{\mathrm{even}}(M, \mathbb{R})$ and $\dim_\tau\colon K_0(B) \to \mathbb{R}$. The proof of Theorem 3.5 shows that $\dim_\tau(\mathrm{ch}(\mathcal{E})) = \mathrm{ch}_\tau(\mathcal{E})$.

3.3. **The projective Dirac operator.** Let (M, g) be a closed oriented Riemannian manifold of even dimension $n > 3$. The oriented frame bundle P_{SO} together with the defining central $\mathbb{Z}/2\mathbb{Z}$-extension $\mathrm{Spin}(n) \to SO(n)$ gives a lifting bundle gerbe $L_{\mathrm{spin}} \to P_{SO}^{[2]}$. Since we assume M to be even-dimensional, the complex spinor module $\Delta_{\mathbb{C}}$ is $\mathbb{Z}/2\mathbb{Z}$-graded, i.e. $\Delta_{\mathbb{C}} = \Delta_+ \oplus \Delta_-$. There is a $\mathbb{Z}/2\mathbb{Z}$-graded twisted vector bundle $S = P_{SO} \times \Delta_{\mathbb{C}}$ with $S_\pm = P_{SO} \times \Delta_\pm$ for L_{spin}, where the action is given by $\gamma([\widehat{g}, \lambda] \otimes v) = \lambda\, \widehat{g} \cdot v$ for $\widehat{g} \in \mathrm{Spin}(n)$, $\lambda \in \mathbb{C}$ and $v \in \Delta_{\mathbb{C}}$. If we fix a connection form $\eta \in \Omega^1(P_{SO}, \mathfrak{so}(n))$, then there is a canonical twisted connection ∇^S on S, which acts like $\nabla^S(\sigma) = d\sigma + \rho_*(\eta) \cdot \sigma$ for a section $\sigma \in C(P_{SO}, S)$, where ρ is induced by $O(n) \to U(n)$.

Let $E \to Y$ be a smooth twisted Hilbert A-module bundle for a bundle gerbe L with $dd(L) = -dd(L_{\text{spin}}) = dd(L_{\text{spin}})$ equipped with a twisted connection ∇^E. Choose a trivialization Q of $L_{\text{spin}} \boxtimes L$, then $Q(S \boxtimes E)$ is a Hilbert A-module bundle over M. Fixing a (flat) connection on Q, $\nabla^{S\boxtimes E} = \nabla^S \otimes \text{id} + \text{id} \otimes \nabla^E$ descends to a connection on $Q(S \boxtimes E)$. The latter bundle carries an action of the complex Clifford bundle $\mathbb{C}\ell(M)$. We have $\nabla^{S\boxtimes E}_X(c \cdot \sigma) = \nabla^{\mathbb{C}\ell(M)}_X(c) \cdot \sigma + c \cdot \nabla^{S\boxtimes E}_X(\sigma)$ for $X \in C^\infty(M, TM)$, $c \in C^\infty(M, \mathbb{C}\ell(M))$, $\sigma \in C^\infty(M, Q(S \boxtimes E))$.

Definition 3.6. Let S, E and Q be as above, then the elliptic first order differential operator $D^E \colon C^\infty(M, Q(S \boxtimes E)) \to C^\infty(M, TM \otimes Q(S \boxtimes E)) \to C^\infty(M, Q(S \boxtimes E))$ where the first map is induced by the connection $\nabla^{S\boxtimes E}$ and the metric and the second is Clifford multiplication will be called the *projective Dirac operator* twisted by E.

The bundle gerbe L_{spin} provides a canonical trivialization Q of $L_{\text{spin}} \boxtimes L_{\text{spin}}$ with the property that $Q(S^* \boxtimes S) \cong \mathbb{C}\ell(M)$ as a bundle of $\mathbb{C}\ell(M)$-$\mathbb{C}\ell(M)$-bimodules. This identifies the operator $D^S \colon C^\infty(M, \mathbb{C}\ell(M)) \to C^\infty(M, \mathbb{C}\ell(M))$ with the right Clifford linear Dirac operator. As described in [3, Section 24.5], D^S yields a class in the K-homology group $KK(C(M, \mathbb{C}\ell(M)), \mathbb{C})$.

Since the connection is even and Clifford multiplication is an odd operation, D^E has the decomposition $D^E_\pm \colon C^\infty(M, Q(S_+ \boxtimes E)) \to C^\infty(M, Q(S_- \boxtimes E))$.

Theorem 3.7. *Let S, E and Q be as in the last paragraph and denote by D^E_+ the positive part of the projective Dirac operator twisted by E, then*

$$\text{ind}(D^E_+) = \int_M \widehat{A}(M)\,\text{ch}(E)\ .$$

In particular, the index of D^E_+ does not depend on the choice of Q.

Proof. Let $\pi_M \colon T^*M \to M$ be the bundle projection and denote by $D(T^*M)$ and $S(T^*M)$ the disc and sphere bundle of the cotangent bundle. The symbol $\sigma \colon \pi_M^* Q(S_+ \boxtimes E) \to \pi_M^* Q(S_- \boxtimes E)$ of D^E_+ yields an element $[Q(S_+ \boxtimes E), Q(S_- \boxtimes E), \sigma] \in K^0_A(D(T^*M), S(T^*M)) \cong K^0_A(T^*M)$. Denote by

$$\phi \colon H^*(M, \mathbb{R}) \to H^{*+\dim(M)}(D(T^*M), S(T^*M); \mathbb{R})$$

the Thom isomorphism. From the Mishchenko-Fomenko index theorem [18] together with Lemma 3.2 we obtain

$$\begin{aligned}\text{ind}(D^E_+) &= \int_M \text{Td}(M)\,\phi^{-1}\left(\text{ch}([Q(S_+ \boxtimes E), Q(S_- \boxtimes E), \sigma])\right) \\ &= \int_M \text{Td}(M)\,\phi^{-1}\left(\text{ch}([S_+, S_-, \sigma_D])\right)\text{ch}(E) = \int_M \widehat{A}(M)\,\text{ch}(E)\end{aligned}$$

where the identification of $\text{Td}(M)\,\phi^{-1}\left(\text{ch}([S_+, S_-, \sigma_D])\right)$ with the $\widehat{A}$-genus described in [16] works in the twisted case as well. □

Theorem 3.8. *Let D be the projective Dirac operator. Let $E \to Y$ be a finitely generated projective twisted Hilbert A-module bundle for a bundle gerbe L with $dd(L) = dd(L_{spin})$. Let Q be a trivialization of $L^* \boxtimes L_{spin}$ and $[E^S] := \rho_Q([E]) = [C(M, Q(S^* \boxtimes E))] \in KK(\mathbb{C}, C(M, \mathbb{C}\ell(M) \otimes A))$. Then*

$[E^S] \otimes_{C(M,\mathbb{C}\ell(M))} [D^S] \in KK(\mathbb{C}, A) \cong K_0(A)$ *is the class representing the Mishchenko-Fomenko index of* D^E.

Proof. It suffices to show that the intersection product coincides with the Fredholm module:

$$\left[L^2(Q(S^* \boxtimes E)), D^E\left(1 + \left(D^E\right)^2\right)^{-\frac{1}{2}}\right] \in KK(\mathbb{C}, A)\ .$$

The proof of this is just a slight modification of [25, Theorem 5.22], therefore we omit it. □

From this decomposition of the index in twisted K-theory, we obtain the following naturality result in analogy to the untwisted case [11, Lemma 3.1].

Corollary 3.9. *Let D be the projective Dirac operator and let $E \to Y$ be a twisted Hilbert A-module bundle for a bundle gerbe L with $dd(L) = dd(L_{spin})$. Given a C^*-algebra homomorphism $\varphi : A \to B$ define the twisted Hilbert B-module bundle F via $F = E \otimes_\varphi B$. Then: $\varphi_*\big(\big[D^E\big]\big) = \big[D^F\big]$, where $\varphi_* : K_0(A) \to K_0(B)$ denotes the induced map on K-theory.*

Proof. Let $[E^S] := [C(M, Q(S^* \boxtimes E))] \in KK(\mathbb{C}, C(M, \mathbb{C}\ell(M) \otimes A))$ and choose a trivialization Q of $L^* \boxtimes L_{\mathrm{spin}}$. The naturality of the Kasparov product yields

$$\begin{aligned}
\varphi_*\big(\big[D^E\big]\big) &= \varphi_*\big([E^S] \otimes_{C(M,\mathbb{C}\ell(M))} [D^S]\big) \\
&= \big((\mathrm{id}_{C(M,\mathbb{C}\ell(M))} \otimes \varphi_*)[C(M, Q(S^* \boxtimes E))]\big) \otimes_{C(M,\mathbb{C}\ell(M))} [D^S] \\
&= [C(M, Q(S^* \boxtimes F))] \otimes_{C(M,\mathbb{C}\ell(M))} [D^S] = \big[D^F\big]\ . \quad \square
\end{aligned}$$

4. Enlargeable manifolds with spin universal cover

In this section we will extend the result of [11, 12] about the Rosenberg index obstruction for closed enlargeable spin manifolds to closed enlargeable orientable manifolds with spin structure on the universal cover. Since $\widetilde{M}$ may be non-compact, we have to work in a twistedly equivariant setting.

Definition 4.1. A connected closed oriented manifold M with fixed metric g is called *enlargeable* if the following holds: For every $\varepsilon > 0$, there is a connected cover $\bar{M} \to M$ carrying a spin structure and an ε-contracting map

$$\bar{\psi}_\varepsilon \colon (\bar{M}, \bar{g}) \to (S^n, g_0)$$

which is constant outside a compact subset of $\bar{M}$ and of nonzero degree. Here, $\bar{g}$ denotes the induced metric on $\bar{M}$ and g_0 is the standard metric on S^n.

Definition 4.2. Let M be a closed even-dimensional oriented Riemannnian manifold and let $\mathcal{V}_{\max}$ be the maximal twisted Mishchenko-Fomenko bundle (Def. 2.16). The *(maximal) twisted Stolz-Rosenberg obstruction* is given by $\theta^{\max}(M) = \mathrm{ind}(D^{\mathcal{V}_{\max}}) \in K_0(C^*_{\max}(\widehat{\pi} \to \pi))$.

4.1. **Almost flat twisted bundles.** In our proof of the non-vanishing of $\theta^{\max}(M)$ for closed enlargeable M with spin universal cover we will extend the argument given in [11, 12] to the twisted setting. Thus, it will rely on almost flat *twisted* bundles.

Definition 4.3. Let $L_{\widehat{\pi}}$ be the Mishchenko-Fomenko bundle gerbe. A sequence $E_i \to \widetilde{M}$, $i \in \mathbb{N}$ of smooth twisted vector bundles for $L_{\widehat{\pi}}$ equipped with connections ∇^i will be called *a sequence of almost flat twisted bundles*, if $\lim_{i\to\infty}\|\Omega_i\| = 0$, where Ω_i is the curvature of the connection ∇^i. The norm is induced by the pointwise norm on $\mathrm{end}(E_i) \to M$ and the maximum norm on the unit sphere bundle in $\Lambda^2(M)$. Moreover, the twistings $\gamma_i^g \colon E_i \to g^*E_i$ considered as sections $C(\widetilde{M}, \mathrm{Hom}(E_i, g^*E_i))$ should be locally Lipschitz continuous with a global Lipschitz constant C independent of i, i.e. each point $\widetilde{m} \in \widetilde{M}$ has an open neighborhood U of $\widetilde{m}$, such that E_i and g^*E_i are trivial over U and such that $\gamma_i^g|_U$ viewed as an element in $C(U, U(V))$ (V being the typical fiber of E_i) satisfies

$$\|\gamma_i^g(\widetilde{m}_1) - \gamma_i^g(\widetilde{m}_2)\| \leq C\, d(\widetilde{m}_1, \widetilde{m}_2)$$

where the metric is induced by the Riemannian structure pulled back to $\widetilde{M}$.

Let M be a closed orientable smooth manifold and let $\bar{M}$ be a cover of M equipped with a spin structure, let $\pi = \pi_1(M)$ and $\bar{\pi} = \pi_1(\bar{M}) \subset \pi$. The bundle projection $P_{\mathrm{Spin}(\bar{M})} \to \bar{M}$ of the principal $\mathrm{Spin}(n)$-bundle induces an isomorphism $\pi_1(P_{\mathrm{Spin}(\bar{M})}) \to \bar{\pi}$, which in turn yields an injective group homomorphism $\sigma \colon \bar{\pi} \to \pi_1(P_{\mathrm{Spin}(\bar{M})}) \to \pi_1(P_{\mathrm{SO}(\bar{M})}) \to \widehat{\pi}$, where the first map is the inverse of the above isomorphism and the other two homomorphism arise from the respective coverings. We call a (set-theoretic) split $s \colon \pi \to \widehat{\pi}$ *compatible*, if $s(gh) = s(g)\sigma(h)$ for $g \in \pi$ and $h \in \bar{\pi}$.

Lemma 4.4. *Let M, $\bar{M}$, π and $\bar{\pi}$ be as in the last paragraph. The choice of a compatible set-theoretic split $s \colon \pi \to \widehat{\pi}$ turns the Hilbert space $H = \ell^2(\pi/\bar{\pi})$ into a projective representation of π, which corresponds to an honest representation of $\widehat{\pi}$.*

Proof. Let $e \in \widehat{\pi}$ be the image of the non-trivial element of $\mathbb{Z}/2\mathbb{Z}$ and let $q = \frac{1}{2}(1 - e) \in \mathbb{C}[\widehat{\pi}]$. Since q is central in the group ring, it yields a well-defined projection $q \colon \ell^2(\widehat{\pi}/\sigma(\bar{\pi})) \to \ell^2(\widehat{\pi}/\sigma(\bar{\pi}))$. The space $q\,\ell^2(\widehat{\pi}/\sigma(\bar{\pi}))$ is a representation of $\widehat{\pi}$, on which e acts by multiplication with -1, i.e. a projective representation of π.

A compatible split now yields an injective map $\pi/\bar{\pi} \to \widehat{\pi}/\sigma(\bar{\pi})$ via $[g] \mapsto [s(g)]$. It induces an isometric isomorphism $\ell^2(\pi/\bar{\pi}) \to q\,\ell^2(\widehat{\pi}/\sigma(\bar{\pi}))$ □

Theorem 4.5. *Let M be an even-dimensional closed orientable manifold that is enlargeable in the sense of Definition 4.1. Let $i \in \mathbb{N}$ be a positive natural number. Then there is a C^*-algebra C_i (which will be constructed in the proof) and a twisted Hilbert C_i-module bundle $E_i \to \widetilde{M}$ for the Mishchenko-Fomenko bundle gerbe $L_{\widehat{\pi}}$ together with a twisted connection ∇_i that has the following properties: The curvature Ω_i of E_i satisfies*

$$\|\Omega_i\| \leq \frac{1}{i}\, C$$

where C is a constant depending only on $\dim(M)$. *Moreover, there is a split extension* $0 \to \mathbb{K} \to C_i \to X_i \to 0$ *with a certain* C^**-algebra* X_i. *In particular, each* $K_0(C_i)$ *splits off a* $\mathbb{Z} = K_0(\mathbb{K})$ *summand and the* $K_0(\mathbb{K})$*-part of the index of the projective Dirac operator* $D_+^{E_i}$ *is different from* 0.

Proof. Let $2n = \dim(M)$ and $\pi = \pi_1(M)$. Since the Chern character is rationally an isomorphism, there is a vector bundle $F \to S^{2n}$ with non-vanishing top Chern class $c_n(F) \neq 0$. Choose a connection η_F on F and fix $i \in \mathbb{N}$. Since M is enlargeable, there exists a spin covering $\bar{M} \to M$ together with a $\frac{1}{i}$-contracting map $\psi\colon \bar{M} \to S^{2n}$, which is constant outside a compact subset K of $\bar{M}$. Let P_F be the principal $U(n)$-bundle of frames in F. Since ψ is constant on $\bar{M}\backslash K$ we can choose a trivialization for the principal $U(n)$-bundle ψ^*P_F over this set:

$$(\psi^*P_F)|_{\bar{M}\backslash K} \cong (\bar{M}\backslash K) \times U(n)$$

such that the pullback connection $\psi^*\eta_F|_{M\backslash K}$ is flat. Let $\rho\colon \bar{M} \to M$ be the covering map, $\widetilde{\rho}\colon \widetilde{M} \to M$ the universal cover and $\bar{\pi} = \pi_1(\bar{M})$. As described in the proof of [12, Proposition 1.5] we can cover M by open sets $U_j, j \in I$, such that each component $V_{\lambda,j} \subset \bar{M}$ of $\rho^{-1}(U_j)$ maps diffeomorphically onto U_j, intersects only one component $V_{\mu,k}$ of $\rho^{-1}(U_k)$ for any k and such that $\psi^*P_F|_{V_{\lambda,j}}$ trivializes. Let $J_j = \pi_0(\rho^{-1}(U_j))$ be the index set labeling the components, likewise set $\widetilde{J}_j = \pi_0(\widetilde{\rho}^{-1}(U_j))$. Let $\widetilde{\varphi}_{\alpha,j}\colon \widetilde{J}_j \to \pi/\bar{\pi}$ be the map that sends αg to $[g] \in \pi/\bar{\pi}$ for $g \in \pi$, where π acts on $\widetilde{J}_j$ by deck transformations. Since $\bar{M} = \widetilde{M}/\bar{\pi}$, this induces bijections $\varphi_{\lambda,j}\colon J_j \to \pi/\bar{\pi}$ for each $\lambda \in J_j$. Note that

$$\varphi_{[\alpha g],j} = g^{-1} \cdot \varphi_{[\alpha],j} \ . \tag{7}$$

Moreover, if $\lambda \in J_j$ and $\mu \in J_k$ belong to components with non-empty intersection, then $\varphi_{\lambda,j}(\kappa) = \varphi_{\mu,k}(\tau)$ if τ and κ intersect.

Consider the Hilbert space $H = \ell^2(\pi/\bar{\pi}) \otimes \mathbb{C}^n$. Let $C_S \subset \mathcal{B}(H)$ be the C^*-algebra generated by the group of all permutations of $\pi/\bar{\pi}$ *and* all multiplications by functions $f\colon \pi/\bar{\pi} \to S^1$. So we have permutation operators with S^1-entries as a generating set of C_S. Let $C_T \subset \mathcal{B}(H)$ be the C^*-algebra generated by linear transformations, which are of the form

$$T\colon H \to H \quad ; \quad T([g] \otimes v) = [h] \otimes T'v \quad \text{and} \quad T|_{([g]\otimes\mathbb{C}^n)^\perp} = 0 \ .$$

for some matrix $T' \in M_n(\mathbb{C})$ and $[g], [h] \in \pi/\bar{\pi}$. Let $C_{S,T}$ be the C^*-algebra generated by C_S and C_T inside of $\mathcal{B}(H)$ and note that C_T is a 2-sided ideal in $C_{S,T}$. Moreover, C_T is isomorphic either to the compact operators or to a matrix algebra. Applying the stabilization trick of [12] we can without loss of generality assume that the former is the case. Let $C_i = \{(c_1, c_2) \in C_{S,T} \times C_{S,T} \mid c_1 - c_2 \in C_T\}$. This algebra fits into a split exact sequence $0 \to C_T \to C_i \to C_{S,T} \to 0$ with the splitting induced by the diagonal map, $C_T \to C_i$ via $a \mapsto (a, 0)$ and $C_i \to C_{S,T}$ via $(a, b) \mapsto b$.

We choose trivializations of ψ^*P_F over the sets $V_{\lambda,j} \subset \bar{M}$, where we take the trivialization fixed above if $V_{\lambda,j}$ is a subset of $\bar{M}\backslash K$. This way we get a cocycle on the double intersections $V_{(\lambda,\mu),(j,k)} = V_{\lambda,j} \cap V_{\mu,k}$:

$$T'_{(\lambda,\mu),(j,k)}\colon V_{(\lambda,\mu),(j,k)} \to U(n) \ .$$

We can extend $T'_{(\lambda,\mu),(j,k)}$ to a cocycle with values in the unitary group $U(C_{S,T})$ as follows:

$$T^1_{(\lambda,\mu),(j,k)}(x)(\varphi_{\lambda,j}(\kappa) \otimes v) = \varphi_{\mu,k}(\tau) \otimes T'_{(\kappa,\tau),(j,k)}(\bar{x})(v) \ ,$$

where $\tau \in \pi/\bar{\pi}$ is the index of the component of $\rho^{-1}(U_k)$ that intersects $V_{\kappa,j}$ and $\bar{x}$ denotes the lift of $\rho(x)$ to the component $V_{\kappa,j}$. This map actually does nothing to the first tensor factor by our previous considerations. Let $T^2_{(\lambda,\mu),(j,k)}$ be the constant map with value $1 \in U(C_{S,T})$. $T'_{(\kappa,\tau),(j,k)}$ is different from the identity only for finitely many pairs (κ, τ). Thus,

$$T_{(\lambda,\mu),(j,k)} \colon V_{(\lambda,\mu),(j,k)} \to U(C_i) \quad ; \quad T_{(\lambda,\mu),(j,k)} = (T^1_{(\lambda,\mu),(j,k)}, T^2_{(\lambda,\mu),(j,k)}) \ .$$

is a well-defined cocycle with values in $U(C_i)$. We therefore get a smooth Hilbert C_i-module bundle $\bar{E}_i \to \bar{M}$, whose pullback to $\widetilde{M}$ will be $E_i = \widetilde{M} \times_M \bar{E}_i$. By Lemma 4.4, the space $\ell^2(\pi/\bar{\pi})$ carries a projective unitary representation of π, which induces a projective representation $r \colon \pi \to U(C_i)$. For $\alpha, \beta \in \widetilde{J}_j$ denote the corresponding components of $\widetilde{\rho}^{-1}(U_j)$ by $W_{\alpha,j}$ and $W_{\beta,j}$ respectively. Let $\lambda = [\alpha]$, $\mu = [\beta]$, $\lambda' = [\alpha g^{-1}]$ and $\mu' = [\beta g^{-1}] \in J_j$. We define

$$\gamma^g \colon W_{\alpha,j} \times C_i \to W_{\alpha g^{-1},j} \times C_i$$

by left multiplication with $r(g)$. Due to equation (7) and with $\varphi_{\lambda,j}(\kappa) = \varphi_{\mu,k}(\tau) = [h] \in \pi/\bar{\pi}$ we have

$$\begin{aligned}\left(T_{(\lambda',\mu'),(j,k)}(x) \cdot r(g)\right)(\varphi_{\lambda,j}(\kappa) \otimes v) &= c_{\widehat{\pi}}(g,h)\, \varphi_{\mu',j}(\tau) \otimes T'_{(\kappa,\tau),(j,k)}(v) \\ &= \left(r(g) \cdot T_{(\lambda,\mu),(j,k)}(x)\right)(\varphi_{\lambda,j}(\kappa) \otimes v) \ .\end{aligned}$$

Thus, γ^g intertwines the transition functions of E_i and $g^*(E_i)$. Therefore it yields a well-defined twisting map $\gamma^g \colon E_i \to g^*(E_i)$. This clearly satisfies the Lipschitz condition, since it even is locally constant.

Let $\eta_{\kappa,j} \in \Omega^1(V_{\kappa,j}, \mathfrak{u}(n))$ be the pullback of η_F via the trivialization. These induce forms in $\Omega^1(V_{\lambda,j}, C^a_{S,T})$, where $C^a_{S,T}$ denotes the anti-selfadjoint operators in $C_{S,T}$, via

$$\left(\eta^{E_i}_{\lambda,j}\right)_x(\xi)(\varphi_{\lambda,j}(\kappa) \otimes v) = \varphi_{\lambda,j}(\kappa) \otimes (\eta_{\kappa,j})_x(\xi) \cdot v \ .$$

Since $\eta_{\kappa,j}$ is non-zero only for finitely many κ, we can extend $\eta^{E_i}_{\lambda,j}$ to a well-defined 1-form with values in the anti-selfadjoint operators of C_i by setting it to zero in the second component. These 1-forms inherit their transformation behaviour from the forms $\eta_{\kappa,j}$. Thus, they yield a C_i-linear connection ∇^i on sections of E_i. Just like above it follows from (7) that ∇^i is a twisted connection. Since the norm of the curvature Ω_i of ∇^i coincides with that of $\psi^*\Omega_F$, we have

$$\|\Omega_i\| = \|\psi^*\Omega_F\| \leq \frac{1}{i}\, C \ .$$

It remains to be shown that the $K_0(\mathbb{K})$-part of $\mathrm{ind}(D^{E_i}_+)$ does not vanish. Here we proceed exactly as in [12]: Let $\mathcal{T} \subset C_T \cong \mathbb{K}$ be the trace class ideal and let D_i be the algebra given by

$$D_i = \{(c_1, c_2) \in C_{S,T} \times C_{S,T} \mid c_1 - c_2 \in \mathcal{T}\} \ .$$

Since the proof of Lemma [12, lemma 2.4] applies to D_i with the changed $C_{S,T}$ as well, D_i is a unital local C^*-algebra with a trace $\tau(c_1, c_2) = \mathrm{tr}(c_1 - c_2)$, which coincides with the trace of the element after projecting it from C_i to $\mathcal{T}$. Its C^*-completion is C_i. Since $K_0(C_i) \cong K_0(D_i)$, we can extend $\dim_\tau$ from Definition 3.4 to a functional on $K_0(C_i)$ and it suffices to prove that $\dim_\tau(\mathrm{ind}(D_+^{E_i})) \neq 0$. The transition functions in the definition of E_i actually take values in $U(D_i)$ and thus lead to a twisted D_i-module bundle $\mathcal{E}_i$ in the sense of Section 3.2. By Theorem 3.7 we have

$$\begin{aligned}\dim_\tau(\mathrm{ind}(D_+^{E_i})) &= \int_M \widehat{A}(M) \dim_\tau(\mathrm{ch}(E_i)) \\ &= \int_M \widehat{A}(M) \dim_\tau(\mathrm{ch}(\mathcal{E}_i)) = \int_M \widehat{A}(M)\, \mathrm{ch}_\tau(\mathcal{E}_i) \ .\end{aligned}$$

We can identify $\Omega_{\mathcal{E}_i} \in \Omega^2(M, \mathrm{end}(\mathcal{E}_i))$ with an equivariant form in $\Omega^2(\widetilde{M}, \mathrm{End}(\mathcal{E}_i))$. If we carry out the integration over a single subset $U_j \subset M$, we could integrate instead over the subset $W_{\alpha,j} \subset \widetilde{M}$ for some $\alpha \in \widetilde{J}_j$. This is independent of the choice of α by equivariance. But over $W_{\alpha,j}$ the form $\tau(\Omega_{\mathcal{E}_i} \wedge \cdots \wedge \Omega_{\mathcal{E}_i})|_{W_{\alpha,j}} \in \Omega^{\mathrm{even}}(W_{\alpha,j}, \mathbb{R})$ coincides with the sum of all $\Omega_{\psi^* F} \wedge \cdots \wedge \Omega_{\psi^* F}|_{V_{\kappa,j}} \in \Omega^{\mathrm{even}}(V_{\kappa,j}, \mathbb{R})$ over $\kappa \in J_j$ by the definition of the trace. Using a partition of unity on M we see that

$$\int_M \widehat{A}(M)\, \mathrm{ch}_\tau(\mathcal{E}_i) = \int_{\bar{M}} \widehat{A}(\bar{M})\, \mathrm{ch}(\psi^* F - \underline{\mathbb{C}^n}) \ ,$$

Since the class of $\mathrm{ch}(\psi^* F - \underline{\mathbb{C}^n})$ is concentrated in degree n we see that the above term is non-vanishing. □

Remark 4.6. Due to the stabilization trick mentioned in the proof the fibers of E_i are isomorphic to $t_i C_i$ for some projection $t_i \in C_i$, where $t_i = 1$ if $\bar{M}$ is non-compact.

Having the sequence E_i of almost flat twisted bundles at hand, we can form the C^*-algebra

$$A = \prod_{i \in \mathbb{N}} C_i$$

of bounded sequences with ith entry in C_i, in which the norm closure of the sequences with only finitely many non-zero entries

$$A' = \overline{\bigoplus_{i \in \mathbb{N}} C_i}^{\|\cdot\|}$$

is a two-sided ideal and we set $Q = A/A'$. Let A_i be the ideal in A consisting of sequences that are 0 everywhere, but in the ith entry.

Theorem 4.7. *There is a smooth twisted Hilbert A-module bundle $E \to \widetilde{M}$ together with a twisted connection*

$$\nabla^E : C^\infty(\widetilde{M}, E) \to C^\infty(\widetilde{M}, T^*\widetilde{M} \otimes E)$$

such that the following holds:

- *$E \cdot A_i$ is isomorphic to E_i as a twisted Hilbert C_i-module bundle.*
- *The connection preserves the subbundles $E \cdot A_i$.*

- *The sequence of curvatures* $\Omega_i \in \Omega^2(M, end(E \cdot A_i))$ *of the connection induced on* $E \cdot A_i$ *by* ∇^E *satisfies* $\lim_{i\to\infty}\|\Omega_i\| = 0$.

Proof. The idea is to see that the product bundle $E_L = \Delta^*_{\widetilde{M}}\left(\prod_{i\in\mathbb{N}} E_i\right)$, where

$$\Delta_{\widetilde{M}} \colon \widetilde{M} \to \prod_{i\in\mathbb{N}} \widetilde{M}$$

is the diagonal map, has locally Lipschitz continuous transition functions. This parallels the construction given in the proof of [11, Lemma 2.1] with the only difference that we have to work equivariantly over $\widetilde{M}$, so we just sketch the differences and refer to [11] for the details: We cover M by subsets U_j, each of them diffeomorphic to I^n, where $I = [0,1]$, such that $\widetilde{M} \to M$ is trivial over U_j via

$$\phi_j \colon U_j \times \pi \to \left.\widetilde{M}\right|_{U_j} .$$

We can find trivializations

$$\psi^1_{i,j} \colon \left.\phi_j^* E_i\right|_{U_j\times\{1\}} \to I^n \times t_i\, C_i$$

of $\left.\phi_j^* E_i\right|_{U_j\times\{1\}}$, such that constant sections of $\phi_j^*E_i$ over $I^k \times \{0,\dots,0\}$ are parallel with respect to ∇_{∂_l} for $1 \leq l \leq k$, where ∇ denotes the connection induced by ∇^{E_i}. Using the twisting we can extend $\psi^1_{i,j}$ to a trivialization $\psi_{i,j}$ of $\left.\phi_j^* E_i\right|_{U_j\times\pi}$ with components $\psi^g_{i,j}$ with $g \in \pi$. Let $\eta^g_{i,j} \in \Omega^1(I^n, t_iC_it_i)$ be the pullback of the connection 1-form of ∇^{E_i}. The way the trivializations are constructed is crucial to prove the estimate given in [11, Lemma 2.3], which now still holds and we have $\|\eta^g_{i,j}\| \leq n \cdot \|\Omega^g_{i,j}\|$, where $\Omega^g_{i,j}$ denotes the curvature of $\eta^g_{i,j}$. The right hand side of the above inequality is independent of $g \in \pi$. Thus, our control of the curvatures carries over to an upper bound on the local connection 1-forms. The trivializations $\psi_{i,j}$ induce transition maps

$$\psi_{i,(j,k)} \colon (U_j \cap U_k) \times \pi \to U(t_iC_it_i)$$

and the upper bound on the local connection 1-forms yields an upper bound on the norm of the derivative $D_{(x,g)}\psi_{i,(j,k)}$ just as described in [11, Lemma 2.5, Proposition 2.6] proving Lipschitz continuity of the transition functions. The Lipschitz condition on the twisting maps ensures that the product of the γ^g_i is continuous, when considered as an element in $C(\widetilde{M}, \mathrm{Hom}(E_L, g^*E_L))$. Thus, E_L is a continuous twisted Hilbert A-module bundle. Note that $\pi = \pi_1(M)$ acts via the adjoint action unitarily on C_i and we set $\mathcal{A} = \widetilde{M} \times_{\mathrm{Ad}} C_i$. E_L corresponds to a projection $t_L \in C(M, \mathcal{A})$, which we can approximate by a projection in $C^\infty(M, \mathcal{A})$ to obtain a smooth twisted Hilbert A-module bundle E by the construction given in Theorem 2.10. We have $E \cdot A_i \cong E_L \cdot A_i = E_i$. The isomorphism may only be continuous, but it can be smoothed.

To construct the connection ∇^E we only need to give local connection 1-forms over the sets $\phi_j(U_j \times \{1\}) \subset \widetilde{M}$ and extend them equivariantly via γ^g to get connection forms over the images of $U_j \times \pi$, which can be patched together with a partition of unity on M. The construction takes the local forms of the E_i and uses a convolution argument to get a smooth form on the product. This is exactly the same as in [11]. □

The twisting γ^g maps the subbundle $E \cdot A'$ into itself, therefore the quotient $\mathcal{W} = E/(E \cdot A')$ is a smooth twisted Hilbert Q-module bundle equipped with a *flat* connection ∇^Q and typical fiber tQ for some projection $t \in Q$. If we now fix a basepoint $\widetilde{m} \in \widetilde{M}$, we get a projective holonomy representation in the sense of Definition 2.18: $(\pi, c_{\widehat{\pi}}) \to \mathrm{End}(\mathcal{W}_{\widetilde{m}}) = tQt$. By the universal property of the maximal twisted group C^*-algebra, this extends to a $*$-homomorphism $\phi\colon C^*_{\max}(\widehat{\pi} \to \pi) \to Q$.

As a consequence of Corollary 3.9, the induced map $\phi_*\colon K_0(C^*_{\max}(\widehat{\pi} \to \pi)) \to K_0(Q)$ maps $\theta^{\max}(M)$ to $\mathrm{ind}(D_+^{\mathcal{W}'})$, where

$$\mathcal{W}' = \widetilde{M} \times tQ = \widetilde{M} \times \mathcal{W}_{\widetilde{m}} \ .$$

Using parallel transport, its equivariance as described before Definition 2.18 with respect to γ^g and flatness of Q we see that $\mathcal{W}'$ is isomorphic to $\mathcal{W}$ as a twisted Hilbert Q-module bundle.

Theorem 4.8. *Let M be a closed smooth orientable even-dimensional manifold with* $\dim(M) \geq 3$ *and $\widetilde{M}$ spin that is enlargeable in the sense of Definition 4.1. Then we have*

$$\theta^{\max}(M) \neq 0 \in K_0(C^*_{\max}(\widehat{\pi} \to \pi)) \ .$$

Proof. As we saw above, we have $\phi_*(\theta^{\max}(M)) = \mathrm{ind}(D_+^{\mathcal{W}}) \in K_0(Q)$ By [12], the group $K_0(Q)$ splits off a summand

$$\prod_{i\in\mathbb{N}} K_0(\mathbb{K}) / \bigoplus_{i\in\mathbb{N}} K_0(\mathbb{K}) \cong \prod_{i\in\mathbb{N}} \mathbb{Z} / \bigoplus_{i\in\mathbb{N}} \mathbb{Z}$$

and the image of $\mathrm{ind}(D_+^{\mathcal{W}})$ in the latter group corresponds to the sequence

$$z_i = \left[p_* \left(\mathrm{ind}(D_+^{E\cdot A_i})\right)\right] = \left[p_* \left(\mathrm{ind}(D_+^{E_i})\right)\right] \ ,$$

where $p\colon C_i \to \mathbb{K}$ is the projection. By Theorem 4.5 it has only non-vanishing entries. □

Remark 4.9. Relaxing the condition about the orientability of M requires incorporating orientation twists of K-theory into the setup, which can be seen as a special case of twisted $\mathbb{Z}/2\mathbb{Z}$-equivariant K-theory as has been observed by Karoubi [14, Remark 6.16], [13]. These can also be described by gerbes (see the Jandl gerbes in [8] and the functor $K_\pm(X)$ in [1], which is naturally equivalent with Karoubi's definition), therefore the above argument generalizes to closed non-orientable manifolds as well. Nevertheless, it seems to be impossible to drop the spin condition for the covers $\bar{M}$ in the definition of enlargeability, since our construction of a projective representation with the right cocycle relies on it.

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